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IDR
as a
Deflation Method

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Program

- Introduction
- Sonneveld spaces
- Inducing dimension reduction
- IDR and deflation

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- Introduction
- Sonneveld spaces
- Inducing dimension reduction
- IDR and deflation

Solve $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$. $\mathbf{A}$ is $n \times n$ non-singular (complex).

Krylov subspace methods

To solve $\mathbf{Ax} = \mathbf{b}$ for $\mathbf{x}$

- **Expansion.**
  Built a Krylov subspace $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$
- **Extraction.**
  Extract an approximate solution $\mathbf{x}_k$ from $\mathcal{K}_k(\mathbf{A}, \mathbf{b})$

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Convergence depends on angle $\mathbf{b}$ and $\mathbf{A}\mathcal{K}_k(\mathbf{A}, \mathbf{b})$

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Convergence depends on spectral properties of $\mathbf{A}$:

– how well eigenvalues are clustered
– on the “conditioning” of the eigenvector basis

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Extraction quality depends on specific method

practical issues as efficiency, stability

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Two ways to improve spectral properties:

- **Preconditioning**.

- **Deflation**.

Krylov subspace methods

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  Remove known components

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Two ways to improve spectral properties:

- **Preconditioning**. To cluster eigenvalues, . . .
  Apply iterative method to $\mathbf{K}^{-1}\mathbf{Ax} = \mathbf{K}^{-1}\mathbf{b}$
- **Deflation**. To replace small eigenvalues by 0
  Remove known components

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$s \in \mathbb{N}$, typically $s = 1, 2, 4, 8$. $\widetilde{\mathbf{R}}$ is a full rank $n \times s$ matrix.

Terminology. $\widetilde{\mathbf{R}}$ is called

the **initial shadow residual** or

the **IDR test matrix**

$s \in \mathbb{N}$, typically $s = 1, 2, 4, 8$. $\widetilde{\mathbf{R}}$ is a full rank $n \times s$ matrix.

Definition.

- Block **Krylov subspace** of order k generated by $\mathbf{A}^*$ and $\widetilde{\mathbf{R}}$

$$\mathcal{K}_k(\mathbf{A}^*, \widetilde{\mathbf{R}}) \equiv \left\{ \sum_{j<k} (\mathbf{A}^*)^j \, \widetilde{\mathbf{R}} \, \gamma_j \mid \gamma_j \in \mathbb{C}^s \right\}$$

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Example.

Bi-CG generates residuals $\mathbf{r}_k^{\text{Bi-CG}} \perp \mathcal{K}_k(\mathbf{A}^*, \widetilde{\mathbf{r}}_0)$,

here $\widetilde{\mathbf{R}} = [\widetilde{\mathbf{r}}_0]$:

$$\mathbf{r}_{k+1}^{\text{Bi-CG}} = \mathbf{r}_k^{\text{Bi-CG}} - \mathbf{A}\mathbf{u}_k \alpha_k \perp \mathcal{K}_{k+1}(\mathbf{A}^*, \widetilde{\mathbf{r}}_0)$$

with $\mathbf{u}_k$ such that $\mathbf{A}\mathbf{u}_k \perp \mathcal{K}_k(\mathbf{A}^*, \widetilde{\mathbf{r}}_0)$.

$s \in \mathbb{N}$, typically $s = 1, 2, 4, 8$. $\widetilde{\mathbf{R}}$ is a full rank $n \times s$ matrix.

Definition. P_k polynomial of exact degree k.

- P_k **Sonneveld subspace** order k generated by $\mathbf{A}$ and $\widetilde{\mathbf{R}}$

$$\mathcal{S}(P_k, \mathbf{A}, \widetilde{\mathbf{R}}) \equiv \left\{ P_k(\mathbf{A})\,\mathbf{v} \mid \mathbf{v} \perp \mathcal{K}_k(\mathbf{A}^*, \widetilde{\mathbf{R}}) \right\}$$

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Example. $\widetilde{\mathbf{R}} = [\,\widetilde{\mathbf{r}}_0\,]$, $\quad \mathbf{r}_k^{\text{Bi-CGSTAB}} = P_k(\mathbf{A})\mathbf{r}_k^{\text{Bi-CG}}$

Note that $\quad \mathbf{r}_k^{\text{Bi-CG}} \perp \mathcal{K}_k(\mathbf{A}^*, \widetilde{\mathbf{r}}_0)$.

In **Bi-CGSTAB**: $\quad P_{k+1}(\lambda) = (1 - \omega_k \lambda) P_k(\lambda)$,

where, with $\mathbf{r}'_k \equiv P_k(\mathbf{A})\mathbf{r}_{k+1}^{\text{Bi-CG}}$,

$$\omega_k \equiv \text{argmin}_{\omega \in \mathbb{C}} \|\mathbf{r}'_k - \omega \mathbf{A}\mathbf{r}'_k\|_2$$

Theorem. $\quad \mathbf{r}_k^{\text{Bi-CGSTAB}} \in \mathcal{S}(P_k, \mathbf{A}, \widetilde{\mathbf{r}}_0)$.

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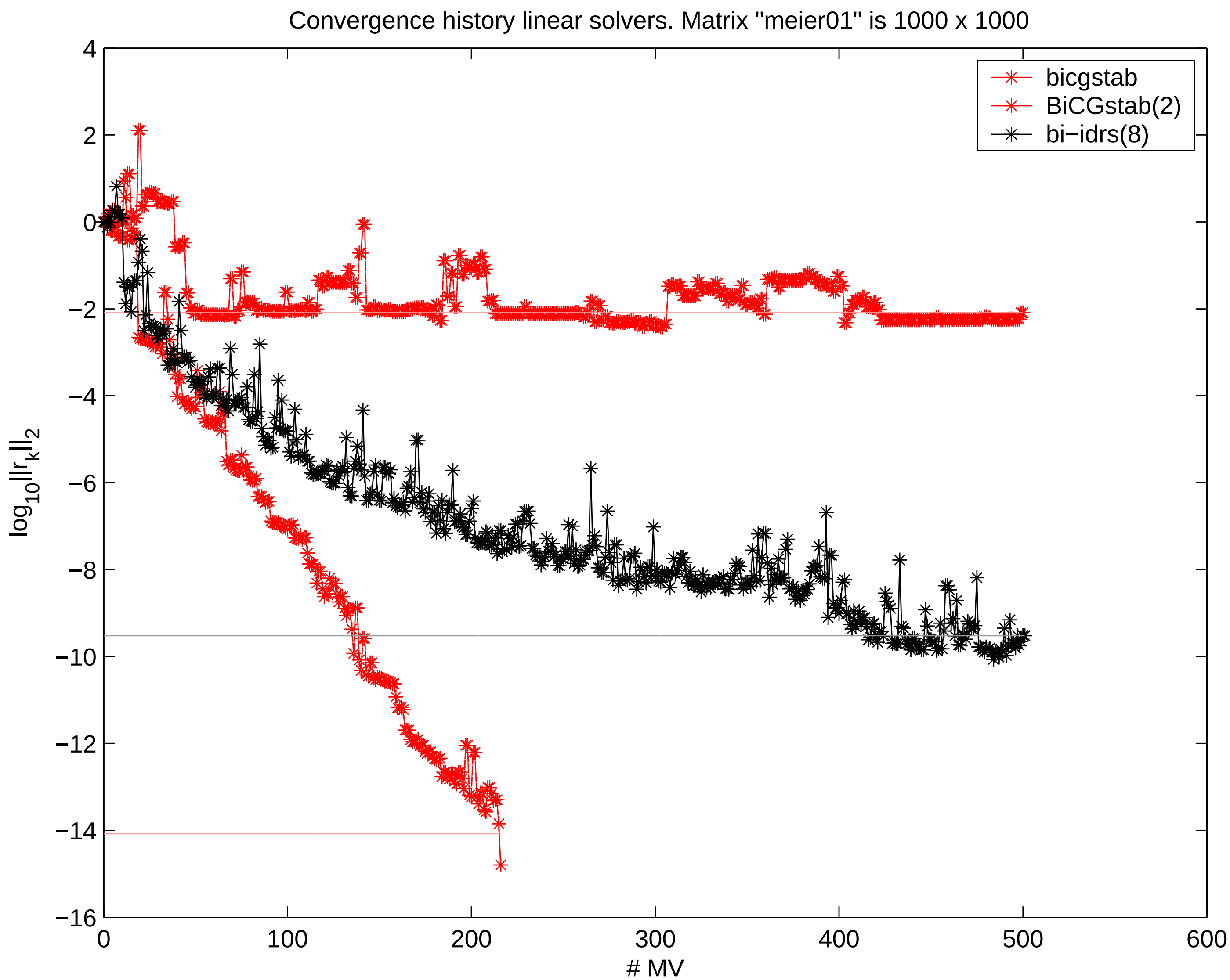
[Sonneveld, van Gijzen, 2008, S. Sonneveld, van Gijzen 2010]

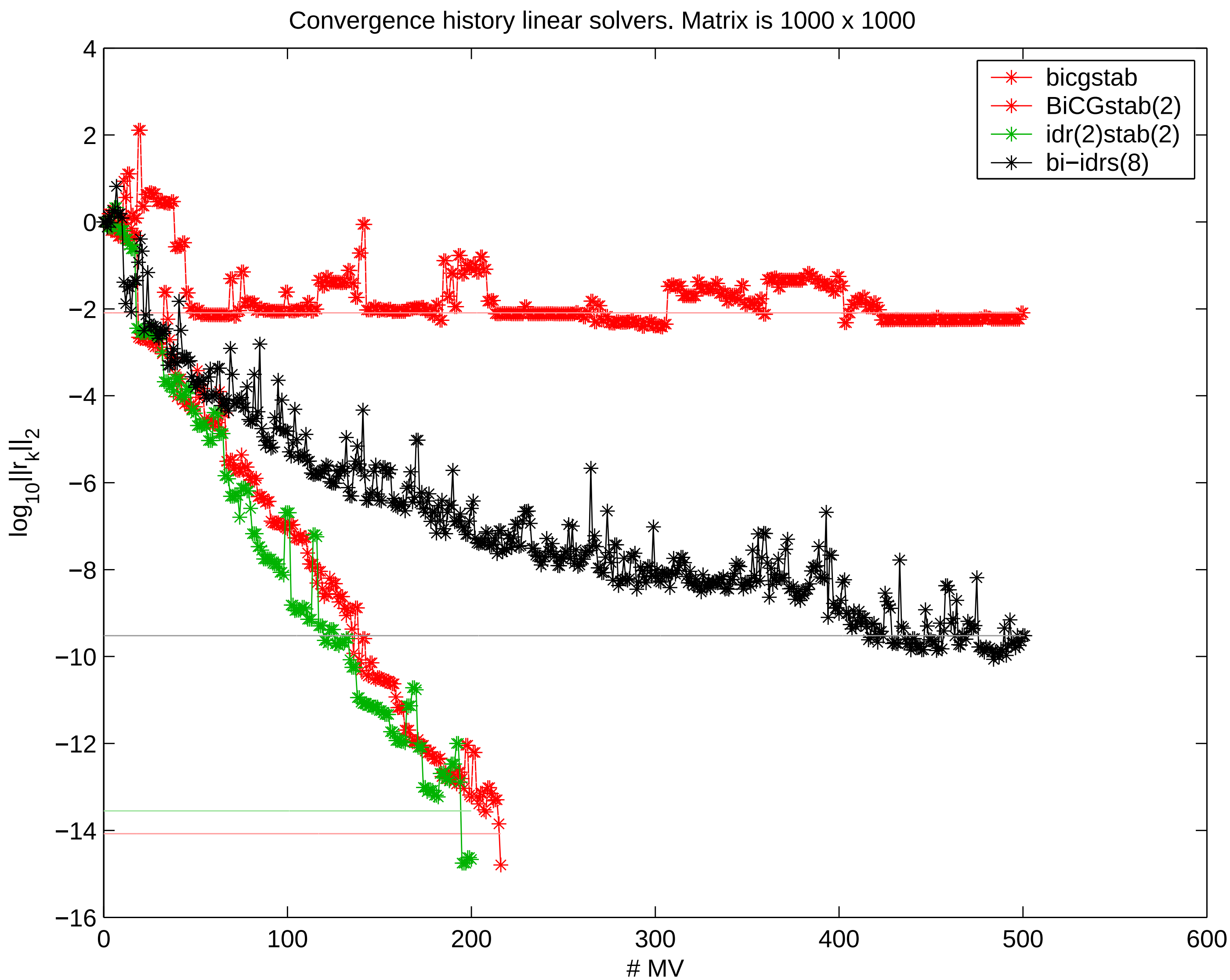
Property. Bi-CGSTAB $\sim$ IDR(s) for $s = 1$ (i.e. $\mathbf{R} = [\tilde{\mathbf{r}}_0]$)



$u_{xx} + u_{yy} + u_{zz} + 1000\, u_x = f$, $\quad f$ is defined by the solution

$$u(x, y, z) = \exp(xyz) \sin(\pi x) \sin(\pi y) \sin(\pi z).$$

Discretized with $(10 \times 10 \times 10)$ volumes. No preconditioning.



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IDR Theorem. With $P_{k+1}(\lambda) \equiv (\alpha - \omega\lambda) P_k(\lambda)$, $\omega \neq 0$

- $(\alpha \mathbf{I} - \omega \mathbf{A}) \left(\mathcal{S}(P_k, \mathbf{A}, \widetilde{\mathbf{R}}) \cap \widetilde{\mathbf{R}}^{\perp} \right) = \mathcal{S}(P_{k+1}, \mathbf{A}, \widetilde{\mathbf{R}})$
- $\mathcal{S}(P_{k+1}, \mathbf{A}, \widetilde{\mathbf{R}}) \subset \mathcal{S}(P_k, \mathbf{A}, \widetilde{\mathbf{R}})$
- $\subsetneq$ if $\mathbf{A}$ has no eigenvector in $\widetilde{\mathbf{R}}^{\perp}$ and $\mathcal{S}(P_k, \mathbf{A}, \widetilde{\mathbf{R}}) \neq \{\mathbf{0}\}$.

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If zeros $P_{k+1} \neq$ eigenvalues **A**, then, increase k leads to
dimension reduction $\mathcal{S}(P_k, \mathbf{A}, \widetilde{\mathbf{R}})$ = dimension increase $\mathcal{K}_k(\mathbf{A}^*, \widetilde{\mathbf{R}})$

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- $\subsetneq$ if $\mathbf{A}$ has no eigenvector in $\widetilde{\mathbf{R}}^{\perp}$ and $\mathcal{S}(P_k, \mathbf{A}, \widetilde{\mathbf{R}}) \neq \{\mathbf{0}\}$.

Corollary $\qquad \mathcal{G}_k \equiv \mathcal{S}(P_k, \mathbf{A}, \widetilde{\mathbf{R}})$, $\qquad \mathcal{G}'_k \equiv \mathcal{G}_k \cap \mathbf{R}^{\perp}$.

With $P_{k+1}(\lambda) \equiv (1 - \omega_k \lambda) P_k(\lambda)$, we have

$$\mathcal{G}_{k+1} \equiv (I - \omega_k \mathbf{A}) \mathcal{G}'_k \subset \mathcal{G}_k \quad \text{and} \quad \mathbf{A}\mathcal{G}'_k \subset \mathcal{G}_k$$

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Induced Dimension Reduction

Select (ω_k), $\widetilde{\mathbf{R}}$. Let $P_{k+1}(\lambda) = (1 - \omega_k \lambda) P_k(\lambda)$.

$\mathcal{G}_0 = \mathbb{C}^n$, $\quad \mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A}) \mathcal{G}'_k \quad$ with $\quad \mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp$.

IDR: construct residuals $\mathbf{r}_k$ in $\mathcal{G}_k$ iteratively by increase k

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IDR: construct residuals $\mathbf{r}_k$ in $\mathcal{G}_k$

The first residual to be in $\mathcal{G}_k$ by construction
is called a **primary residual**.

The implementation may rely on other residuals:
secondary residuals

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IDR: construct residuals $\mathbf{r}_k$ in $\mathcal{G}_k$

All our updates for residuals are of the form

$$\mathbf{r} \leftarrow \mathbf{r} - \alpha \mathbf{c} \quad \text{with} \quad \mathbf{c} = \mathbf{A}\mathbf{u} \quad \textbf{and u available.}$$

Hence, $\mathbf{x} \leftarrow \mathbf{x} + \alpha \mathbf{u}$. Updates $\mathbf{r}$ and $\mathbf{x}$ are **consistent**.

Approximate solution x gets a(n almost) free ride:

only vector updates, no MVs, no inner products.

Focuss on **r**

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IDR: $\mathbf{r}_k \in \mathcal{G}_k$, then construct residual $\mathbf{r}_{k+1}$ in $\mathcal{G}_{k+1}$

	IDR step		pol. step	
$\mathcal{G}_k$	$\longrightarrow$	$\mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp$	$\longrightarrow$	$\mathcal{G}_{k+1} \equiv (\mathbf{I}-\omega_k\mathbf{A})\mathcal{G}'_k$
	Π_1		$\mathbf{I}-\omega_k\mathbf{A}$	
$\mathbf{r}_k$		$\mathbf{r}'_k = \Pi_1\mathbf{r}_k$		$\mathbf{r}_{k+1} = \mathbf{r}'_k - \omega_k\mathbf{A}\mathbf{r}'_k$

Π_1 is a skew projection that projects onto $\widetilde{\mathbf{R}}^\perp$

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$\Pi_1 = \mathbf{I} - \mathbf{V}\sigma^{-1}\widetilde{\mathbf{R}}^*$ with $\sigma \equiv \widetilde{\mathbf{R}}^*\mathbf{V}$ $\; s \times s$ non-singular,

and $\mathbf{V} \equiv \mathbf{V}_k$ is $n \times s$ matrix with $\quad$ span$(\mathbf{V}) \subset \mathcal{G}_k$

$[\mathbf{r}_k, \mathbf{V}_k]$ full rank

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$[\mathbf{r}_k, \mathbf{V}_k]$ full rank

$\Pi_1(\mathbf{v}_j) = 0$ for the columns of $\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_s]$

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$\mathbf{V}_k$	Π_1	$\mathbf{V}'_k =????$	$\mathbf{I}-\omega_k\mathbf{A}$	$\mathbf{V}_{k+1} = \mathbf{V}'_k - \omega_k\mathbf{A}\mathbf{V}'_k$

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Π_1 is a skew projection that projects onto $\widetilde{\mathbf{R}}^\perp$

$$\mathbf{w} \in \mathcal{G}'_k \quad \Rightarrow \quad \mathbf{A}\mathbf{w} \in \mathcal{G}_k \quad \Rightarrow \quad \Pi_1\mathbf{A}\mathbf{w} \in \mathcal{G}'_k$$

Induced Dimension Reduction

Select (ω_k), $\widetilde{\mathbf{R}}$. Let $P_{k+1}(\lambda) = (1-\omega_k\lambda)P_k(\lambda)$.

$\mathcal{G}_0 = \mathbb{C}^n$, $\quad \mathcal{G}_{k+1} \equiv (\mathbf{I}-\omega_k\mathbf{A})\mathcal{G}'_k \quad$ with $\quad \mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp$.

IDR: construct residuals $\mathbf{r}_{k+1}$ in $\mathcal{G}_{k+1}$, $\mathbf{V}_{k+1}$ span in $\mathcal{G}_{k+1}$.

	IDR step		pol. step	
$\mathcal{G}_k$	$\longrightarrow$	$\mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}$	$\longrightarrow$	$\mathcal{G}_{k+1} \equiv (\mathbf{I}-\omega_k\mathbf{A})\mathcal{G}'_k$
	Π_1		$\mathbf{I}-\omega_k\mathbf{A}$	
$\mathbf{V}_k$		$\mathbf{V}'_k = ????$		$\mathbf{V}_{k+1} = \mathbf{V}'_k - \omega_k\mathbf{A}\mathbf{V}'_k$

Π_1 is a skew projection that projects onto $\widetilde{\mathbf{R}}^\perp$

$$\mathbf{w} \in \mathcal{G}'_k \quad \Rightarrow \quad \mathbf{A}\mathbf{w} \in \mathcal{G}_k \quad \Rightarrow \quad \Pi_1\mathbf{A}\mathbf{w} \in \mathcal{G}'_k$$

$$\Rightarrow \qquad \mathcal{K}_{s+1}(\Pi_1\mathbf{A}, \mathbf{r}'_k) \subset \mathcal{G}'_k$$

Induced Dimension Reduction

Select (ω_k), $\widetilde{\mathbf{R}}$. Let $P_{k+1}(\lambda) = (1 - \omega_k \lambda) P_k(\lambda)$.

$\mathcal{G}_0 = \mathbb{C}^n, \quad \mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A}) \mathcal{G}'_k$ with $\mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp$.

IDR: construct residuals $\mathbf{r}_{k+1}$ in $\mathcal{G}_{k+1}$, $\mathbf{V}_{k+1}$ span in $\mathcal{G}_{k+1}$.

	IDR step		pol. step	
$\mathcal{G}_k$	$\longrightarrow$	$\mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}$	$\longrightarrow$	$\mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A}) \mathcal{G}'_k$
$\mathbf{V}_k$	Π_1	$\mathbf{V}'_k = ????$	$\mathbf{I} - \omega_k \mathbf{A}$	$\mathbf{V}_{k+1} = \mathbf{V}'_k - \omega_k \mathbf{A} \mathbf{V}'_k$

Π_1 is a skew projection that projects onto $\widetilde{\mathbf{R}}^\perp$

$$\mathbf{w} \in \mathcal{G}'_k \quad \Rightarrow \quad \mathbf{A}\mathbf{w} \in \mathcal{G}_k \quad \Rightarrow \quad \Pi_1 \mathbf{A}\mathbf{w} \in \mathcal{G}'_k$$

$$\Rightarrow \quad \mathcal{K}_{s+1}(\Pi_1 \mathbf{A}, \mathbf{r}'_k) \subset \mathcal{G}'_k$$

- $\mathbf{V}'_k$ with $\text{span}(\mathbf{r}'_k, \mathbf{V}'_k) = \mathcal{K}_{s+1}(\Pi_1 \mathbf{A}, \mathbf{r}'_k)$

Induced Dimension Reduction

Select (ω_k), $\widetilde{\mathbf{R}}$. Let $P_{k+1}(\lambda) = (1 - \omega_k\lambda)P_k(\lambda)$.

$\mathcal{G}_0 = \mathbb{C}^n$, $\quad \mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k\mathbf{A})\mathcal{G}'_k \quad$ with $\quad \mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp$.

IDR: construct residuals $\mathbf{r}_{k+1}$ in $\mathcal{G}_{k+1}$, $\mathbf{V}_{k+1}$ span in $\mathcal{G}_{k+1}$.

	IDR step		pol. step	
$\mathcal{G}_k$	$\longrightarrow$	$\mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}$	$\longrightarrow$	$\mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k\mathbf{A})\mathcal{G}'_k$
	Π_1		$\mathbf{I} - \omega_k\mathbf{A}$	
$\mathbf{V}_k$		$\mathbf{V}'_k =????$		$\mathbf{V}_{k+1} = \mathbf{V}'_k - \omega_k\mathbf{A}\mathbf{V}'_k$

Π_1 is a skew projection that projects onto $\widetilde{\mathbf{R}}^\perp$

$$\mathbf{w} \in \mathcal{G}'_k \quad \Rightarrow \quad \mathbf{A}\mathbf{w} \in \mathcal{G}_k \quad \Rightarrow \quad \Pi_1\mathbf{A}\mathbf{w} \in \mathcal{G}'_k$$

$$\Rightarrow \quad \mathcal{K}_{s+1}(\Pi_1\mathbf{A}, \mathbf{r}'_k) \subset \mathcal{G}'_k$$

- $\mathbf{V}'_k$ with $\mathrm{span}(\mathbf{r}'_k, \mathbf{V}'_k) = \mathcal{K}_{s+1}(\Pi_1\mathbf{A}, \mathbf{r}'_k)$; $\quad \mathbf{A}\mathbf{V}'_k$ is side product

Induced Dimension Reduction

Select (ω_k), $\widetilde{\mathbf{R}}$. Let $P_{k+1}(\lambda) = (1-\omega_k\lambda)P_k(\lambda)$.

$\mathcal{G}_0 = \mathbb{C}^n, \quad \mathcal{G}_{k+1} \equiv (\mathbf{I}-\omega_k\mathbf{A})\mathcal{G}'_k \quad$ with $\quad \mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp$.

IDR: construct residuals $\mathbf{r}_{k+1}$ in $\mathcal{G}_{k+1}$, $\mathbf{V}_{k+1}$ span in $\mathcal{G}_{k+1}$.

$$\begin{array}{ccccc} & \text{IDR step} & & \text{pol. step} & \\ \mathcal{G}_k & \longrightarrow & \mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}} & \longrightarrow & \mathcal{G}_{k+1} \equiv (\mathbf{I}-\omega_k\mathbf{A})\mathcal{G}'_k \\ & \Pi_1 & & \mathbf{I}-\omega_k\mathbf{A} & \\ \mathbf{V}_k & & \mathbf{V}'_k = !! & & \mathbf{V}_{k+1} = \mathbf{V}'_k - \omega_k\mathbf{A}\mathbf{V}'_k \end{array}$$

Theorem. Assume $\quad \mathrm{span}([\mathbf{r}_k, \mathbf{V}_k]) \subset \mathcal{G}_k$.

If $[\mathbf{r}'_k, \mathbf{V}'_k]$ spans $\mathcal{K}_{s+1}(\Pi_1\mathbf{A}, \mathbf{r}'_k)$ and no break-down occurs,

then $\quad \mathrm{span}([\mathbf{r}_{k+1}, \mathbf{V}_{k+1}]) \subset \mathcal{G}_{k+1}$.

This is essentially the only way to move to the next $\mathcal{G}_k$

$[\mathbf{r}'_k, \mathbf{V}'_k]$ spans $\mathcal{K}_{s+1}(\Pi_1 \mathbf{A}, \mathbf{r}'_k)$. How to select the basis $\mathbf{V}_k$?

(In exact arithmetic)

any basis leads to the same projection.

Program

- Introduction
- Sonneveld spaces
- Inducing dimension reduction
- IDR and deflation

IDR step		pol. step	
$\mathcal{G}_k$	$\underset{\Pi_1}{\longrightarrow}$	$\mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp$	$\underset{\mathbf{I}-\omega_k\mathbf{A}}{\longrightarrow}$	$\mathcal{G}_{k+1} \equiv (\mathbf{I}-\omega_k\mathbf{A})\mathcal{G}'_k$
$\mathbf{r}_k$		$\mathbf{r}'_k = \Pi_1 \mathbf{r}_k$		$\mathbf{r}_{k+1} = \mathbf{r}'_k - \omega_k \mathbf{A}\mathbf{r}'_k$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$ with $\sigma \equiv \mathbf{R}^*\mathbf{A}\mathbf{U}$ and $\mathbf{V} = \mathbf{A}\mathbf{U}$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{Q}$ with $\mathbf{Q} \equiv \mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$

$$\begin{array}{ccccc} & \text{IDR step} & & \text{pol. step} & \\ \mathcal{G}_k & \longrightarrow & \mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^{\perp} & \longrightarrow & \mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A})\mathcal{G}'_k \\ & \Pi_1 & & \mathbf{I} - \omega_k \mathbf{A} & \\ \mathbf{r}_k & & \mathbf{r}'_k = \Pi_1 \mathbf{r}_k & & \mathbf{r}_{k+1} = \mathbf{r}'_k - \omega_k \mathbf{A}\mathbf{r}'_k \end{array}$$

$\Pi_1 = \mathbf{I} - \mathbf{AU}\sigma^{-1}\widetilde{\mathbf{R}}^*$ with $\sigma \equiv \mathbf{R}^*\mathbf{AU}$ and $\mathbf{V} = \mathbf{AU}$

$\Pi_1 = \mathbf{I} - \mathbf{AQ}$ with $\mathbf{Q} \equiv \mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$

Deflated system:

$$\text{Solve} \quad \Pi_1 \mathbf{A}\mathbf{x}' = \mathbf{r}'_k \quad \text{for} \quad \mathbf{x}' \perp \widetilde{\mathbf{R}}. \qquad (*)$$

Then $\mathbf{x} = \mathbf{x}_k + \mathbf{Q}\mathbf{r}_k + (\mathbf{I} - \mathbf{QA})\mathbf{x}'$ solves $\mathbf{Ax} = \mathbf{b}$.

Note. $\Pi_1 \mathbf{A} : \widetilde{\mathbf{R}}^{\perp} \to \widetilde{\mathbf{R}}^{\perp}$ and $\Pi_1 \mathbf{r}_k \in \widetilde{\mathbf{R}}^{\perp}$.

$\mathcal{K}_k(\Pi_1 \mathbf{A}, \Pi_1 \mathbf{r}_k)$ leads to approximate solutions of $(*)$ in $\widetilde{\mathbf{R}}^{\perp}$.

$$\begin{array}{ccccc} & \text{IDR step} & & \text{pol. step} & \\ \mathcal{G}_k & \longrightarrow & \mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp & \longrightarrow & \mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A})\mathcal{G}'_k \\ & \Pi_1 & & \mathbf{I} - \omega_k \mathbf{A} & \\ \mathbf{r}_k & & \mathbf{r}'_k = \Pi_1 \mathbf{r}_k & & \mathbf{r}_{k+1} = \mathbf{r}'_k - \omega_k \mathbf{A} \mathbf{r}'_k \end{array}$$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$ with $\sigma \equiv \mathbf{R}^*\mathbf{A}\mathbf{U}$ and $\mathbf{V} = \mathbf{A}\mathbf{U}$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{Q}$ with $\mathbf{Q} \equiv \mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$

Deflated system:

$$\text{Solve} \quad \Pi_1 \mathbf{A}\mathbf{x}' = \mathbf{r}'_k \quad \text{for} \quad \mathbf{x}' \perp \widetilde{\mathbf{R}}. \qquad (*)$$

Then $\mathbf{x} = \mathbf{x}_k + \mathbf{Q}\mathbf{r}_k + (\mathbf{I} - \mathbf{Q}\mathbf{A})\mathbf{x}'$ solves $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Note. $\Pi_1 \mathbf{A} : \widetilde{\mathbf{R}}^\perp \to \widetilde{\mathbf{R}}^\perp$ and $\Pi_1 \mathbf{r}_k \in \widetilde{\mathbf{R}}^\perp$.

$\mathcal{K}_k(\Pi_1\mathbf{A}, \Pi_1\mathbf{r}_k)$ leads to approximate solutions of $(*)$ in $\widetilde{\mathbf{R}}^\perp$.

Solve $(*)$ with s steps of some Krylov method:

Advantage. At the same time

a basis $\mathbf{V}'$ of $\mathcal{K}_{s+1}(\Pi_1\mathbf{A}, \Pi_1\mathbf{r}_k)$ and a smaller residual.

$$\begin{array}{ccccc} & \text{IDR step} & & \text{pol. step} & \\ \mathcal{G}_k & \longrightarrow & \mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp & \longrightarrow & \mathcal{G}_{k+1} \equiv (\mathbf{I}-\omega_k\mathbf{A})\mathcal{G}'_k \\ \mathbf{r}_k & \Pi_1 & \mathbf{r}'_k = \Pi_1 \mathbf{r}_k & \mathbf{I}-\omega_k\mathbf{A} & \mathbf{r}_{k+1} = \mathbf{r}'_k - \omega_k \mathbf{A}\mathbf{r}'_k \end{array}$$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$ with $\sigma \equiv \mathbf{R}^*\mathbf{A}\mathbf{U}$ and $\mathbf{V} = \mathbf{A}\mathbf{U}$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{Q}$ with $\mathbf{Q} \equiv \mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$

Deflated system:

$$\text{Solve } \Pi_1 \mathbf{A}\mathbf{x}' = \mathbf{r}'_k \text{ for } \mathbf{x}' \perp \widetilde{\mathbf{R}}. \qquad (*)$$

Then $\mathbf{x} = \mathbf{x}_k + \mathbf{Q}\mathbf{r}_k + (\mathbf{I} - \mathbf{Q}\mathbf{A})\mathbf{x}'$ solves $\mathbf{A}\mathbf{x} = \mathbf{b}$.

Note. $\Pi_1\mathbf{A} : \widetilde{\mathbf{R}}^\perp \to \widetilde{\mathbf{R}}^\perp$ and $\Pi_1 \mathbf{r}_k \in \widetilde{\mathbf{R}}^\perp$.

$\mathcal{K}_k(\Pi_1\mathbf{A}, \Pi_1\mathbf{r}_k)$ leads to approximate solutions of $(*)$ in $\widetilde{\mathbf{R}}^\perp$.

However, if $\mathbf{s}_s$ is an s-step Krylov residual, then

$$\Pi_1(\mathbf{s}_s - \omega_k \mathbf{A}\mathbf{s}_s) = \mathbf{r}_{k+1}.$$

$$\begin{array}{ccccc} & \text{IDR step} & & \text{pol. step} & \\ \mathcal{G}_k & \longrightarrow & \mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^{\perp} & \longrightarrow & \mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A})\mathcal{G}'_k \\ & \Pi_1 & & \mathbf{I} - \omega_k \mathbf{A} & \\ \mathbf{r}_k & & \mathbf{r}'_k = \Pi_1 \mathbf{r}_k & & \mathbf{r}_{k+1} = \mathbf{r}'_k - \omega_k \mathbf{A} \mathbf{r}'_k \end{array}$$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$ with $\sigma \equiv \mathbf{R}^*\mathbf{A}\mathbf{U}$ and $\mathbf{V} = \mathbf{A}\mathbf{U}$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{Q}$ with $\mathbf{Q} \equiv \mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$

Interpretation. Solve $\mathbf{A}\mathbf{u} = \mathbf{r}$

Search for an approximate solution in span($\mathbf{U}$)

$$\mathbf{R}^*\mathbf{A}\mathbf{U}\alpha = \mathbf{R}^*\mathbf{r}$$

Then $\quad \mathbf{u} \approx \mathbf{U}\alpha = \mathbf{U}(\mathbf{R}^*\mathbf{A}\mathbf{U})^{-1}\mathbf{R}^*\mathbf{r} = \mathbf{Q}\mathbf{r}$

$$\begin{array}{ccccc} & \text{IDR step} & & \text{pol. step} & \\ \mathcal{G}_k & \longrightarrow & \mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp & \longrightarrow & \mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A})\mathcal{G}'_k \\ & \Pi_1 & & \mathbf{I} - \omega_k \mathbf{A} & \\ \mathbf{r}_k & & \mathbf{r}'_k = \Pi_1 \mathbf{r}_k & & \mathbf{r}_{k+1} = \mathbf{r}'_k - \omega_k \mathbf{A} \mathbf{r}'_k \end{array}$$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$ with $\sigma \equiv \mathbf{R}^*\mathbf{A}\mathbf{U}$ and $\mathbf{V} = \mathbf{A}\mathbf{U}$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{Q}$ with $\mathbf{Q} \equiv \mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$

Interpretation. Solve $\mathbf{A}\mathbf{u} = \mathbf{r}$

Search for an approximate solution in span($\mathbf{U}$)

$$\mathbf{R}^*\mathbf{A}\mathbf{U}\alpha = \mathbf{R}^*\mathbf{r}$$

Then $\quad \mathbf{u} \approx \mathbf{U}\alpha = \mathbf{U}(\mathbf{R}^*\mathbf{A}\mathbf{U})^{-1}\mathbf{R}^*\mathbf{r} = \mathbf{Q}\mathbf{r}$

In multigrid:

- $\mathbf{R}^*$ represents the restriction operator
- $\mathbf{U}$ represents the prolongation
- $\mathbf{Q}$ is the coarse grid correction

$$\begin{array}{ccccc} & \text{IDR step} & & \text{pol. step} & \\ \mathcal{G}_k & \longrightarrow & \mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp & \longrightarrow & \mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A})\mathcal{G}'_k \\ & \Pi_1 & & \mathbf{I} - \omega_k \mathbf{A} & \\ \mathbf{r}_k & & \mathbf{r}'_k = \Pi_1 \mathbf{r}_k & & \mathbf{r}_{k+1} = \mathbf{r}'_k - \omega_k \mathbf{A} \mathbf{r}'_k \end{array}$$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$ with $\sigma \equiv \mathbf{R}^*\mathbf{A}\mathbf{U}$ and $\mathbf{V} = \mathbf{A}\mathbf{U}$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{Q}$ with $\mathbf{Q} \equiv \mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$

Interpretation. Solve $\mathbf{A}\mathbf{u} = \mathbf{r}$

Search for an approximate solution in span($\mathbf{U}$)

$$\mathbf{R}^*\mathbf{A}\mathbf{U}\alpha = \mathbf{R}^*\mathbf{r}$$

Then $\quad \mathbf{u} \approx \mathbf{U}\alpha = \mathbf{U}(\mathbf{R}^*\mathbf{A}\mathbf{U})^{-1}\mathbf{R}^*\mathbf{r} = \mathbf{Q}\mathbf{r}$

Then the update of the residual $\mathbf{r}$ is

$$\mathbf{r}'_k = \mathbf{r}_k - \mathbf{A}\mathbf{Q}\mathbf{r}_k$$

with update of the approximate solution

$$\mathbf{x}'_k = \mathbf{x}_k + \mathbf{Q}\mathbf{r}_k$$

$$\begin{array}{ccccc} & \text{IDR step} & & \text{pol. step} & \\ \mathcal{G}_k & \longrightarrow & \mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp & \longrightarrow & \mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A})\mathcal{G}'_k \\ & \Pi_1 & & \mathbf{I} - \omega_k \mathbf{A} & \\ \mathbf{r}_k & & \mathbf{r}'_k = \Pi_1 \mathbf{r}_k & & \mathbf{r}_{k+1} = \mathbf{r}'_k - \omega_k \mathbf{A}\mathbf{r}'_k \end{array}$$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$ with $\sigma \equiv \mathbf{R}^*\mathbf{A}\mathbf{U}$ and $\mathbf{V} = \mathbf{A}\mathbf{U}$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{Q}$ with $\mathbf{Q} \equiv \mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$

Interpretation.

Coarse grid correction: $\mathbf{r}'_k = \mathbf{r}_k - \mathbf{A}\mathbf{Q}\mathbf{r}_k$ and $\mathbf{x}'_k = \mathbf{x}_k + \mathbf{Q}\mathbf{r}_k$

Smoothing: $\mathbf{r}_{k+1} = \mathbf{r}'_k - \omega \mathbf{A}\mathbf{r}'_k$ and $\mathbf{x}_{k+1} = \mathbf{x}'_k + \omega \mathbf{r}'_k$

	IDR step		pol. step	
$\mathcal{G}_k$	$\longrightarrow$	$\mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp$	$\longrightarrow$	$\mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A})\mathcal{G}'_k$
$\mathbf{r}_k$	Π_1	$\mathbf{r}'_k = \Pi_1 \mathbf{r}_k$	$\mathbf{I} - \omega_k \mathbf{A}$	$\mathbf{r}_{k+1} = \mathbf{r}'_k - \omega_k \mathbf{A}\mathbf{r}'_k$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$ with $\sigma \equiv \mathbf{R}^*\mathbf{A}\mathbf{U}$ and $\mathbf{V} = \mathbf{A}\mathbf{U}$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{Q}$ with $\mathbf{Q} \equiv \mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$

Interpretation.

Coarse grid correction: $\mathbf{r}'_k = \mathbf{r}_k - \mathbf{A}\mathbf{Q}\mathbf{r}_k$ and $\mathbf{x}'_k = \mathbf{x}_k + \mathbf{Q}\mathbf{r}_k$

Smoothing: $\mathbf{r}_{k+1} = \mathbf{r}'_k - \omega\mathbf{A}\mathbf{r}'_k$ and $\mathbf{x}_{k+1} = \mathbf{x}'_k + \omega\mathbf{r}'_k$

Combine: $\mathbf{r}_{k+1} = (\mathbf{I} - \omega\mathbf{A})\Pi_1\mathbf{r}_k = \mathbf{r}_k - \mathbf{A}(\mathbf{Q} + \omega\Pi_1)\mathbf{r}$

$\mathbf{x}_{k+1} = \mathbf{x}_k + (\mathbf{Q} + \omega\Pi_1)\mathbf{r}_k$

With $\mathbf{P} \equiv \mathbf{Q} + \omega\Pi$,

$\mathbf{I} - \mathbf{A}\mathbf{P}$ is the **residual reduction operator**

IDR step		pol. step	
$\mathcal{G}_k$	$\longrightarrow$	$\mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp$	$\longrightarrow$	$\mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A})\mathcal{G}'_k$
$\mathbf{r}_k$	Π_1	$\mathbf{r}'_k = \Pi_1 \mathbf{r}_k$	$\mathbf{I} - \omega_k \mathbf{A}$	$\mathbf{r}_{k+1} = \mathbf{r}'_k - \omega_k \mathbf{A}\mathbf{r}'_k$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$ with $\sigma \equiv \mathbf{R}^*\mathbf{A}\mathbf{U}$ and $\mathbf{V} = \mathbf{A}\mathbf{U}$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{Q}$ with $\mathbf{Q} \equiv \mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$

Interpretation.

Coarse grid correction: $\mathbf{r}'_k = \mathbf{r}_k - \mathbf{A}\mathbf{Q}\mathbf{r}_k$ and $\mathbf{x}'_k = \mathbf{x}_k + \mathbf{Q}\mathbf{r}_k$

Smoothing: $\mathbf{r}_{k+1} = \mathbf{r}'_k - \omega \mathbf{A}\mathbf{r}'_k$ and $\mathbf{x}_{k+1} = \mathbf{x}'_k + \omega \mathbf{r}'_k$

Combine: $\mathbf{r}_{k+1} = (\mathbf{I} - \omega\mathbf{A})\Pi_1 \mathbf{r}_k = \mathbf{r}_k - \mathbf{A}(\mathbf{Q} + \omega\Pi_1)\mathbf{r}$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + (\mathbf{Q} + \omega\Pi_1)\mathbf{r}_k$$

With $\mathbf{P} \equiv \mathbf{Q} + \omega\Pi$,

$\mathbf{I} - \mathbf{P}\mathbf{A}$ is the **error reduction operator**

	IDR step		pol. step	
$\mathcal{G}_k$	$\longrightarrow$	$\mathcal{G}'_k \equiv \mathcal{G}_k \cap \widetilde{\mathbf{R}}^\perp$	$\longrightarrow$	$\mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A})\mathcal{G}'_k$
$\mathbf{r}_k$	Π_1	$\mathbf{r}'_k = \Pi_1 \mathbf{r}_k$	$\mathbf{I} - \omega_k \mathbf{A}$	$\mathbf{r}_{k+1} = \mathbf{r}'_k - \omega_k \mathbf{A}\mathbf{r}'_k$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$ with $\sigma \equiv \mathbf{R}^*\mathbf{A}\mathbf{U}$ and $\mathbf{V} = \mathbf{A}\mathbf{U}$

$\Pi_1 = \mathbf{I} - \mathbf{A}\mathbf{Q}$ with $\mathbf{Q} \equiv \mathbf{U}\sigma^{-1}\widetilde{\mathbf{R}}^*$

Interpretation.

Coarse grid correction: $\mathbf{r}'_k = \mathbf{r}_k - \mathbf{A}\mathbf{Q}\mathbf{r}_k$ and $\mathbf{x}'_k = \mathbf{x}_k + \mathbf{Q}\mathbf{r}_k$

Smoothing: $\mathbf{r}_{k+1} = \mathbf{r}'_k - \omega\mathbf{A}\mathbf{r}'_k$ and $\mathbf{x}_{k+1} = \mathbf{x}'_k + \omega\mathbf{r}'_k$

Combine: $\mathbf{r}_{k+1} = (\mathbf{I} - \omega\mathbf{A})\Pi_1\mathbf{r}_k = \mathbf{r}_k - \mathbf{A}(\mathbf{Q} + \omega\Pi_1)\mathbf{r}$

$$\mathbf{x}_{k+1} = \mathbf{x}_k + (\mathbf{Q} + \omega\Pi_1)\mathbf{r}_k$$

With $\mathbf{P} \equiv \mathbf{Q} + \omega\Pi$,

$\mathbf{I} - \mathbf{P}\mathbf{A}$ is the **error reduction operator**

IDR can be viewed as

Richardson with a flexible preconditioner

Spectrum of IDR's error reduction operator

$$\Lambda(\mathbf{I} - \mathbf{PA}) = \Lambda(\mathbf{I} - \mathbf{AP}) = \Lambda(\Pi_1(\mathbf{I} - \omega\mathbf{A}))$$

The eigenvalues of the IDR error reduction operator

$$\mathbf{I} - \mathbf{PA}$$

are related to the eigenvalues of the IDR deflated matrix

$$\Pi_1\mathbf{A}.$$

[Erlangga, Nabben, 2008]

Theorem. For a $\lambda \in \mathbb{C}, \lambda \neq 0$ we have that

if $\Pi_1\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ then $\Pi_1(\mathbf{I} - \omega\mathbf{A})\mathbf{v} = (1 - \omega\lambda)\mathbf{v}$

If $\Pi_1\mathbf{A}\mathbf{v} = \mathbf{0}$ then $(\mathbf{I} - \mathbf{PA})\mathbf{v} = \mathbf{0}$.

The spectrum of IDR's deflated operator

With (ω_k), $\quad P_{k+1}(\lambda) = (1 - \omega_k \lambda) P_k(\lambda)$,

$$\mathcal{G}_{k+1} \equiv (\mathbf{I} - \omega_k \mathbf{A}) \mathcal{G}'_k \quad \text{with} \quad \mathcal{G}'_k \equiv \mathcal{G}_k \cap \mathbf{R}^\perp,$$

Let $\mathbf{V}$ be such that $\text{span}(\mathbf{V}) \subset \mathcal{G}_k$ as in IDR

The spectrum of IDR's deflated operator

With (ω_k), $\quad P_{k+1}(\lambda) = (1-\omega_k\lambda)P_k(\lambda)$,

$$\mathcal{G}_{k+1} \equiv (\mathbf{I}-\omega_k\mathbf{A})\mathcal{G}'_k \quad \text{with} \quad \mathcal{G}'_k \equiv \mathcal{G}_k \cap \mathbf{R}^\perp,$$

Let $\mathbf{V}$ be such that $\text{span}(\mathbf{V}) \subset \mathcal{G}_k$

Theorem

- 0 is an eigenvalue of $\Pi_1\mathbf{A}$ with geometric multiplicity $\geq s$.
- If $P_k(\mu) = 0$ (i.e., $\mu = 1/\omega_j$ for some $j < k$), then
  μ is an eigenvalue of $\Pi_1\mathbf{A}$ with geometric multiplicity $\geq s$
- If $P_k(\mu) = P_k(\mu) = \ldots = P_k^{(\ell-1)}(\mu) = 0$ then
  μ is an eigenvalue of $\Pi_1\mathbf{A}$ with algebraic multiplicity $\geq \ell s$

If $\text{span}(\mathbf{V}) \subset \mathcal{G}_k$,
then $\mathbf{V} = P_k(\mathbf{A})\underline{\mathbf{V}}$ for some $n \times s$ matrix $\underline{\mathbf{V}} \perp \mathcal{K}_k(\mathbf{A}^*, \widetilde{\mathbf{R}})$.

- If $\widetilde{\mathbf{R}}^*(\mathbf{A}-\lambda\mathbf{I})^{-1}\underline{\mathbf{V}}$ is singular,
  then λ is an eigenvalue of $\Pi_1\mathbf{A}$.

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Theorem. $\underline{\mathbf{V}}$ is independent of P_k in IDR.

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Comment. $\mathbf{V}_k$ only spans an s-dimensional subspace.
(with expected deflation as indicated by the s-fold eig. 0)

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Though $\mathbf{V}_k$ has a "memory"
of all ks vectors in the preceeding $\mathbf{V}_j$ $(j < k)$:
it allows control (clustering) of ks eigenvalues.

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Comment. When selecting $\omega_j = 1$ $(j \leq k)$,

0 is an eigenvalue of $(\mathbf{I}-\omega_k\mathbf{A})\Pi_1$ with multiplicity $(k+1)s$.

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  0 is an eigenvalue of $(\mathbf{I}-\omega_k\mathbf{A})\Pi_1$ with multiplicity s.
- If $\mathbf{V}_k$ is constructed with the IDR scheme, then
  the remaining $n-(k+1)s$ eigenvalues
  do not depend on ω_j, $j < k$.

Conclusions

- IDR is a [class of] very effective methods
- IDR can be viewed as a deflation method with a flexible preconditioner
- The deflated matrices as produced in IDR have remarkable spectral properties
- It is not clear how to exploit these elegant relations and properties to explain the effectiveness of IDR

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